THREE-DIMENSIONAL ANALOG OF A VORTICAL CHAPLYGIN COLUMN (A GENERALIZED HILL VORTEX)

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In connection with the possibility of the practical application of annular vortices in the struggle against atmospheric pollution in industrial enterprises [1], there is rising interest in cylindrical and spherical vortices.

There is given below a generalization of a spherical Hill vortex, which is a three-dimensional analog of a vortical Chaplygin column [2]. As a partial case, a detailed investigation is made of a spherical helical vortex: inside of such a vortex, the motion of the vortex is an axisymmetric homogeneous helical flow. A picture of the flow inside this vortex is constructed, and it is shown that, similarly to a vortical Chaplygin column or a Hill vortex, it can move translationally at a constant rate in a liquid which is at rest at infinity. The limiting rate of motion of a helical vortex, determined by the requirement that the pressure within it remain positive, depends on the pressure of the liquid at infinity and on its density. This velocity is approximately two times less than the corresponding velocity of a Chaplygin column and four times less than the velocity of a Hill vortex.

Although, for the practical use of spherical vortices, a whole series of factors must still be taken into consideration, above all the viscosity, the fact is of interest that within the framework of a model of an ideal liquid a class of spherical vortices can be constructed, one of whose partial cases is a Hill vortex.

1. In the general case of an axisymmetric vortical flow of an incompressible nonviscous liquid, the equation for the flow function ψ in spherical coordinates R, θ, φ has the form [3]

$$\frac{\partial^2\psi}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) + \Phi(\psi)\Phi'(\psi) + F'(\psi)R^2\sin^2\theta = 0 \qquad (1.1)$$

Here the function $\Phi(\psi)$ determines the law of change of the φ-component of the velocity, while the function $F(\psi)$ characterizes the law of the distribution of the energy in the flow.

Let us investigate the question of the possibility of the existence of a spherical vortex, moving translationally in a liquid which is at rest at infinity. Inside of the vortex, in a system of coordinates moving together with the vortex, the flow is described by the differential equation (1.1). Imparting to the whole mass of the liquid a velocity equal in value and opposite to the direction of the velocity of the vortex, we go over to a form which is more convenient for consideration of the problem of potential flow around a spherical vortex. Under these circumstances, the continuity of the corresponding components of the velocity at the surface of the vortex is required. Let us pass on to consideration of one of the problems of potential and vortical flows [1]. As is remarked in [1, 4], in general form, this problem has not been investigated, even for $\Phi(\psi) = 0$ and the simplest functions $F(\psi)$. Only with $\Phi(\psi) = 0$ and $F(\psi) = A\psi$, where A is a constant, is there an exact solution, i.e., the flow function is a "very simple flow field, known as a spherical Hill vortex" [4].

Leaving unchanged the law of distribution of the energy, and assuming, in addition, that $\Phi(\psi) = k\psi$, we write Eq. (1.1) in the form

$$\frac{\partial^2\psi}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) + k^2\psi - AR^2\sin^2\theta = 0 \qquad (1.2)$$

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The flow function ψ_+ for potential flow around a sphere of radius a has the form

$$\psi_+ = {}^1/_2\, UR^2 (1 - a^3 / R^3) \sin^2\theta \qquad (R > a) \quad . \tag{1.3}$$

Thus, the problem reduces to seeking a flow function ψ_- inside the vortex, satisfying Eq. (1.2) and the flowing conditions at the surface of the vortex:

$$\psi_- |_{R=a} = 0 \tag{1.4}$$

$$\left.\frac{\partial\psi_-}{\partial R}\right|_{R=a} = \left.\frac{\partial\psi_+}{\partial R}\right|_{R=a} = {}^3/_2\, Ua \sin^2\theta \quad , \tag{1.5}$$

2. Let us consider the generalization of a Hill vortex. We shall seek the solution of the problem (1.2), (1.4), (1.5) in the form

$$\psi_- = f(R) \sin^2\theta \qquad (R < a) \tag{2.1}$$

where $f(R)$ is an arbitrary differentiable function of the variable R. This leads to an ordinary differential equation,

$$R^2 f''(R) + (k^2R^2 - 2) f(R) = AR^4 \quad . \tag{2.2}$$

To find the general solution of this equation with $A \neq 0$, we go over to the new variables z and w(z):

$$f = Ak^{-7/2}R^{1/2}w(z), \qquad z = kR \quad .$$

We obtain an inhomogeneous Bessel equation;

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - {}^9/_4)\, w = z^{7/2} \quad .$$

A general solution of this equation, stable at all endpoints inside the sphere, has the form

$$w = CJ_{3/2}(z) + s_{5/2,\, 3/2}(z)$$

where C is an arbitrary constant; $J_{3/2}(z)$ is a Bessel function of order $^3/_2$; $S_{5/2,\, 3/2}(z)$ is a Lommel function.

From the theory of Bessel functions, we have $S_{\nu+1,\nu}(z) = z^\nu$; specifically, $S_{5/2,\, 3/2}(z) = z^{3/2}$.

Taking this into consideration, we obtain

$$w(z) = C\, J_{3/2}(z) + z^{3/2} \quad .$$

Going back to the previous variables, we find

$$f(R) = Ak^{-7/2}R^{1/2} (CJ_{3/2}(kR) + (kR)^{3/2}) \tag{2.3}$$

$$\psi_- = Ak^{-7/2}R^{1/2} (CJ_{3/2}(kR) + (kR)^{3/2}) \sin^2\theta \quad .$$

We assume, first of all, that $0 < k < b/a$, where $b = 4.4934$ is the least positive root of the function $J_{3/2}(z)$.

Then from the condition (1.4)

$$C = -(ka)^{3/2} / J_{3/2}(ka)$$

and, consequently,

$$\psi_- = Ak^{-2}\left(1 - \left(\frac{a}{R}\right)^{3/2} \frac{J_{3/2}(kR)}{J_{3/2}(ka)}\right) R^2 \sin^2\theta \quad . \tag{2.4}$$

Satisfying condition (1.5) we find

$$A = {}^3/_2\, Uk^2 \frac{J_{3/2}(ka)}{3J_{3/2}(ka) - ka\, J_{1/2}(ka)} \quad .$$

Thus,

$$\psi_- = {}^3/_2\, U \frac{J_{3/2}(ka)}{3J_{3/2}(ka) - kaJ_{1/2}(ka)} \left(1 - \left(\frac{a}{R}\right)^{3/2} \frac{J_{3/2}(kR)}{J_{3/2}(ka)}\right) R^2 \sin^2\theta \quad . \tag{2.5}$$

The flow under consideration has the following special characteristic: the vector-vortex at each point of the surface of the sphere is located in a tangential plane passing through this point, and inclined at exactly the same angle to the corresponding meridional plane.

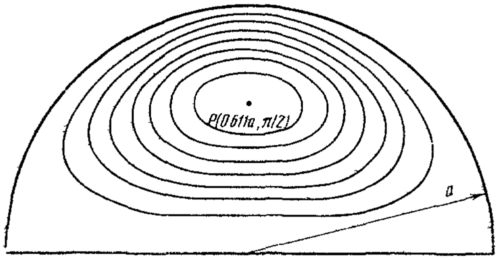

Fig. 1

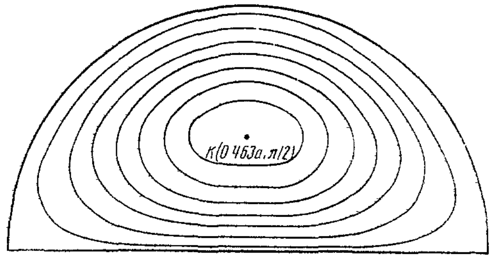

Fig. 2

The components of the vector-vortex Ω [3] are

$$\Omega_R = k v_R, \quad \Omega_\theta = k v_\theta, \quad \Omega_\varphi = k v_\varphi - AR \sin\theta .$$

At the surface of the sphere

$$\Omega_R = 0, \quad \Omega_\theta = {}^3/_2\, k U \sin\theta, \quad \Omega_\varphi = -Aa \sin\theta .$$

Therefore,

$$\chi = \Omega_\varphi / \Omega_\theta = \frac{\lambda J_{3/2}(\lambda)}{\lambda J_{1/2}(\lambda) - 3J_{3/2}(\lambda)} = \text{const} \qquad (\lambda = ka) \tag{2.6}$$

which proves the assertion made above.

Thus, the parameter k is determined by the angle of inclination of the vector-vortex at a given point of the surface of the sphere to the corresponding meridional plane.

Let us consider the limiting cases $k=0$ and $k=b/a$.

Using expressions for the Bessel functions with small values of the argument, it can be verified that, in the first case, at the surface of the sphere $\Omega_\theta = 0$, and the flow function (2.5) determines the flow inside a spherical Hill vortex. In the second case, we obtain a spherical vortex, at whose surface $\Omega_\psi = 0$; inside this vortex, the flow of the liquid is homogeneous ($k =$ const) and helical. We shall call such a vortex a spherical helical vortex. Following [3], we shall call the parameter k the intensity of the homogeneous spherical motion; in the case under consideration, it is determined by the ratio b/a.

3. Let us consider a spherical helical vortex. Setting $k = b/a$ in (2.5), we find

$$\psi_- = {}^3/_2\, U \frac{a^{3/2}}{b J_{1/2}(b)} R^{1/2} J_{3/2}(bR/a) \sin^2\theta . \tag{3.1}$$

Since $J_{1/2}(b) < 0$, and $J_{3/2}(bR/a) > 0$ with $R < a$, then $\psi_- < 0$, $\psi_+ > 0$. With $k > b/a$, the flow function would change sign inside of the sphere.

The equation of the lines of flow in meridional cross sections of the sphere ($R < a$) is

$$\left(\frac{R}{a}\right)^{1/2} J_{3/2}\left(b\frac{R}{a}\right) \sin^2\theta = \text{const} . \tag{3.2}$$

These lines are shown in Fig. 1 for equally-spaced values of ψ_-. They include the separating lines of the flow: the axis of symmetry and meridional line of the sphere.

The maximum absolute value for the values of ψ^- is reached at the point $(0.611a, \pi/2)$, around which the lines of the flow are constricted. At this point, the components of the velocity $v_R = v_\theta = 0$. For purposes of comparison, we note that in the case of a vortical Chaplygin column [2] the corresponding point is located at a distance $\delta = 0.48a$ from the axis of symmetry of the flow, where a is the radius of the cylindrical vortex; in the case of a Hill vortex, its coordinate is $(0.707a, \pi/2)$. In the last two cases, this point is critical; through it passes the motionless central vortical filament rectilinear in the case of a Chaplygin column, and circular in a Hill vortex).

In the case under consideration, the point of an extremum of the function ψ_- is not critical, since at this point the azimuthal component of the velocity v_φ differs from zero. Therefore, the vortical line passing through the given point, and which constitutes a neighborhood in a plane perpendicular to the axis of symmetry of the flow, rotates around the latter with a certain peripheral velocity. This velocity can be found from the formula

$$v_\varphi = \frac{k\psi}{R\sin\theta} = {}^3/_2\, U \left(\frac{a}{R}\right)^{1/2} \frac{J_{3/2}(bR/a)}{J_{1/2}(b)} \sin\theta . \tag{3.3}$$

Carrying out the corresponding calculations using tables of Bessel functions with $R/a=0.611$ and $\theta=\pi/2$, we obtain

$$v_\varphi = -2.87U .$$

The greatest absolute value of the φ-component of the velocity is attained at the point $(0.463a, \pi/2)$

$$|v_{\varphi\max}| = 3.35U .$$

Lines of the intersection of the level surfaces of the function $v_\varphi(R, \theta)$ and the plane of a meridional cross section of the sphere are shown in Fig. 2.

The remaining components of the velocity can be found from the formulas

$$v_R = \frac{1}{R\sin\theta}\frac{\partial\psi}{R\partial\theta} = \frac{3}{bJ_{1/2}(b)}U\left(\frac{a}{R}\right)^{3/2} J_{3/2}\left(b\frac{R}{a}\right)\cos\theta \tag{3.4}$$

$$v_\theta = -\frac{1}{R\sin\theta}\frac{\partial\psi}{\partial R} = -\frac{3}{2bJ_{1/2}(b)}U\left(\frac{a}{R}\right)^{3/2}\left(b\frac{R}{a}J_{1/2}\left(b\frac{R}{a}\right) - J_{3/2}\left(b\frac{R}{a}\right)\right)\sin\theta . \tag{3.5}$$

The maximal value of the velocity inside the vortex arises at its center:

$$|v_{\max}| = \left(\frac{2b}{\pi}\right)^{1/2}\frac{1}{|J_{1/2}(b)|}U = 5.3U .$$

This same velocity inside a Chaplygin column [2] is 2.48U and inside a Hill vortex, 1.5U. This points to a greater intensity of the vortical motion inside a spherical helical vortex.

Let us pass on to determination of the pressure inside the vortex. As was shown in [7], with the helical motion of a nonviscous liquid, the reserve of energy in the whole mass of liquid is constant, i.e., the Bernoulli equation is applicable to the flow as a whole:

$$\frac{p}{\rho} + \frac{v_R^2 + v_\theta^2 + v_\varphi^2}{2} = \text{const} . \tag{3.6}$$

With a transition through the boundary of the vortex, the pressure changes continuously. Therefore, we can set

$$\text{const} = p_0/\rho + U^2/2$$

where p_0 is the pressure at infinity.

Since the greatest absolute value of the velocity is attained at the center of the vortex, the lowest pressure will exist at this point:

$$p_{\min} = p_0 + \left(1 - \frac{2b}{\pi J_{1/2}^2(b)}\right)\frac{\rho U^2}{2} = p_0 - 13.5\rho U^2 . \tag{3.7}$$

For the pressure inside the vortex to remain positive the following condition must be observed:

$$U < 0.272\,(p_0/\rho)^{1/2} . \tag{3.8}$$

With exactly the same ratio p_0/ρ, the limiting velocity of this motion is approximately two times less than the corresponding velocity of a Chaplygin column and four times less than the velocity of a Hill vortex.

Imparting to the whole mass of the liquid a velocity U in an opposite direction, we obtain a spherical helical vortex, moving translationally in a liquid which is at rest at infinity.

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